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Elementary remarks on Ulam–Hyers stability of linear functional equations

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Abstract

In this paper the general method for proving stability of linear functional equations is described.
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1. Introduction

In the last years a great number of papers dealing with the Ulam–Hyers stability of functional equations have been published. Many of them treat functional equations in several variables and what is given is an inequality in several variables and the first step in order to prove stability consists in certain manipulations transforming that inequality in several variables in an inequality in one variable (see, for instance, [1–9,13–39]).

In the general case the manipulations come down to the following: there is (in an appropriate framework) a functional equation

$$E_1(F) = E_2(F)$$

in the unknown function F and appear several variables (note that F is a one-place function); moreover we have a function f satisfying the related inequality

$$d(E_1(f), E_2(f)) \leq \Delta,$$

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where Δ is a certain function depending (or not) on the variables involved (d is a distance).

After a certain number of manipulations in the inequality, only one variable remains and we get something of this form

$$d(H\{f[G(x)]\}, f(x)) \leq \delta(x). \quad (1)$$

At this moment, under suitable conditions, the standard procedure described in [10] can be applied to get a solution F of the functional equation

$$H\{F[G(x)]\} = F(x)$$

which is *near* the function f . To conclude the stability result it is then necessary to show that the function obtained is indeed a solution of the original equation $E_1(F) = E_2(F)$ and this part strongly depends on the form of the functional equation involved, on the set S and on the space X .

Throughout this paper we consider the family of linear functional equations of the form

$$\sum_{i=1}^s b_i F\left(\sum_{k=1}^m a_{ik} x_k\right) = 0 \quad (2)$$

and the related inequalities

$$\left| \sum_{i=1}^s b_i f\left(\sum_{k=1}^m a_{ik} x_k\right) \right| \leq \delta, \quad (3)$$

where $F, f: S \rightarrow X$, S is a vector space over a field \mathbb{K} of characteristic zero, X is a complex Banach space, whose norm is denoted by $|\cdot|$, b_1, b_2, \dots, b_s are nonzero complex numbers with $B := \sum_{i=1}^s b_i \neq 0$, a_{ik} , $i = 1, 2, \dots, s$, $k = 1, 2, \dots, m$, are in \mathbb{K} , x_k , $k = 1, 2, \dots, m$, are elements of S . For sake of simplicity, we assume δ to be a positive constant.

The stability of a great number of equations which are special cases of (2) has been proved in the last years, but the procedures for transforming the inequality (3) into form (1) appear as ad hoc substitutions invented just for the specific case.

Herein we intend to present this procedure in a general form.

2. Notations and preliminaries

By introducing the following notations:

$$\mathbf{a}_i = (a_{i1}, a_{i2}, \dots, a_{im}), \quad \mathbf{x} = (x_1, x_2, \dots, x_m) \quad \text{and} \quad \langle \mathbf{a}_i, \mathbf{x} \rangle = \sum_{k=1}^m a_{ik} x_k,$$

we write Eq. (2) as

$$\sum_{i=1}^s b_i F(\langle \mathbf{a}_i, \mathbf{x} \rangle) = 0 \quad (4)$$

and inequality (3) as

$$\left| \sum_{i=1}^s b_i f(\langle \mathbf{a}_i, \mathbf{x} \rangle) \right| \leq \delta. \quad (5)$$

Setting $\mathbf{x} = 0$ in (4), we have $F(0) \sum_{i=1}^s b_i = 0$. Since $B := \sum_{i=1}^s b_i \neq 0$, this forces $F(0) = 0$.

Setting $\mathbf{x} = 0$ in (5) we get $|Bf(0)| \leq \delta$ and this gives $|f(0)| \leq \frac{\delta}{|B|}$.

The problem we intend to address is the following: *does there exist a one-dimensional subspace of S^m such that on it the inequality (5) “generates” an inequality of the form*

$$|hf(gz) - f(z)| \leq \bar{\delta}(\delta, |B|), \quad (6)$$

where $z \in S$, $g \in \mathbb{K}$ and h is a complex number with $|h| \neq 1$?

In case of affirmative answer, we have the following possibilities:

- (i) $g = 0$ or $g = 1$ or $h = 0$: in all these cases inequality (6) implies the boundedness of the function f and consequently the stability, since the function identically zero is a solution of (4);
- (ii) g different from 0 and 1 and h different from 0.

In this last case, assuming, for instance, $|h| < 1$, we can apply the method described in [10] to obtain a function

$$F(z) = \lim_{n \rightarrow +\infty} h^n f(g^n z), \quad \text{such that} \quad |F(z) - f(z)| \leq \frac{\bar{\delta}}{1 - |h|}, \quad z \in \mathbb{C}.$$

Substituting in (5) \mathbf{x} with $g^n \mathbf{x}$ and multiplying f by h^n , we have

$$\left| \sum_{i=1}^s h^n b_i f(g^n(\mathbf{a}_i, \mathbf{x})) \right| \leq |h|^n \delta$$

and, for $n \rightarrow +\infty$,

$$\sum_{i=1}^s b_i F\left(\sum_{k=1}^m a_{ik} x_k\right) = 0,$$

i.e., F is a solution of (4).

Note that starting from two different inequalities of the form (6), i.e.,

$$|h_1 f(g_1 z) - f(z)| \leq \bar{\delta}_1(\delta, |B|), \quad |h_2 f(g_2 z) - f(z)| \leq \bar{\delta}_2(\delta, |B|), \quad (7)$$

both in the condition (ii) with $|h_1| < 1$ and $|h_2| < 1$, we obtain the same solution of (4).

Indeed, assume we have two solutions F_1 and F_2 of Eq. (4) such that

$$\begin{aligned} |F_1(z) - f(z)| &\leq \sigma_1, & F_1(z) &= \lim_{n \rightarrow +\infty} h_1^n f(g_1^n z), \\ |F_2(z) - f(z)| &\leq \sigma_2, & F_2(z) &= \lim_{n \rightarrow +\infty} h_2^n f(g_2^n z). \end{aligned}$$

From inequalities (7), by multiplying by h_2^{n-1} and substituting z with $g_2^{n-1} z$, we obtain

$$|h_1 h_2^{n-1} f(g_1 g_2^{n-1} z) - h_2 f(g_2 z)| \leq |h_2|^{n-1} (\bar{\delta}_1 + \bar{\delta}_2)$$

and, taking the limit as $n \rightarrow +\infty$, we have

$$F_2(z) = \lim_{n \rightarrow +\infty} h_1 h_2^{n-1} f(g_2^{n-1} g_1 z) = h_1 \lim_{n \rightarrow +\infty} h_2^{n-1} f(g_2^{n-1} g_1 z) = h_1 F_2(g_1 z).$$

Thus, taking the limit as $n \rightarrow +\infty$ in the following inequality

$$|F_2(z) - h_1^n f(g_1^n z)| = |h_1^n F_2(g_1^n z) - h_1^n f(g_1^n z)| = |h_1|^n |F_2(g_1^n z) - f(g_1^n z)| \leq |h_1|^n \bar{\delta}_2,$$

we arrive to

$$F_2(z) = \lim_{n \rightarrow +\infty} h_1^n f(g_1^n z) = F_1(z).$$

If $|h| > 1$ we can write (6) in the form

$$\left| f(gz) - \frac{1}{h} f(z) \right| \leq \frac{\bar{\delta}}{|h|},$$

then by setting $gz = u$, $z = \frac{u}{g}$, we proceed in the same way.

3. Results

Denote by A the $s \times m$ matrix with entries a_{ik} and consider the linear system

$$A\mathbf{y} = \mathbf{v}, \quad (8)$$

where the vector $\mathbf{v} = [v_1, v_2, \dots, v_s] \in \mathbb{K}^s$ has the following form:

- there exists a family $\{S_\ell\}$, $\ell = 1, 2, \dots, p$, of nonempty disjoint subsets of $\{1, 2, \dots, s\}$ and p different nonzero elements of \mathbb{K} , $\gamma_1, \gamma_2, \dots, \gamma_p$ such that for $i \in S_\ell$, $v_i = \gamma_\ell$;
- if $\bigcup_{\ell=1}^p S_\ell \neq \{1, 2, \dots, s\}$, for $i \in S_0 := \{1, 2, \dots, s\} \setminus \bigcup_{\ell=1}^p S_\ell$, $v_i = 0$; set $B_0 := \sum_{i \in S_0} b_i$;
- at least one and at most two of the numbers $B_\ell := \sum_{i \in S_\ell} b_i$, $\ell = 1, 2, \dots, p$, are different from zero.

Without loss of generality, we may assume that $B_1 \neq 0$ and $B_3 = \dots = B_p = 0$.

If, for some \mathbf{v} having the previous form the system (8) has a solution $\mathbf{y} = [y_1, y_2, \dots, y_m] \in \mathbb{K}^m$, then by substituting in (5) \mathbf{x} with $w\mathbf{y}$, $w \in S$, we obtain the following inequality:

$$|B_1 f(\gamma_1 w) + B_2 f(\gamma_2 w)| \leq \delta + |B_0 f(0)| \leq \delta \left(1 + \frac{|B_0|}{|B|} \right).$$

By setting $w = \frac{z}{\gamma_1}$ and $g = \frac{\gamma_2}{\gamma_1}$ we have

$$|B_1 f(z) + B_2 f(gz)| \leq \delta \left(1 + \frac{|B_0|}{|B|} \right).$$

If B_2 is different from zero we have the inequality (6) where $h = -\frac{B_2}{B_1}$, $\bar{\delta} = \frac{\delta + |B_0 f(0)|}{|B_1|}$.

Thus, if $|h| \neq 1$ we are done. If $|h| = 1$, we do not have any information about stability.

If $B_2 = 0$ we simply conclude that f is bounded.

Also taking in account the quite ample arbitrariness of \mathbf{v} , in general we cannot expect that system (8) has a solution which gives the stability.

To see this, consider the following functional equation (where S is \mathbb{R} or \mathbb{C}) (see [28], where it is called Euler–Lagrange equation):

$$F(x_1 + x_2 + x_3) + F(x_1 - x_2 + x_3) + F(x_1 + x_2 - x_3) + F(x_1 - x_2 - x_3) - 4[F(x_1) + F(x_2) + F(x_3)] = 0. \quad (9)$$

The system (8) becomes the following:

$$\begin{cases} y_1 + y_2 + y_3 = v_1, \\ y_1 - y_2 + y_3 = v_2, \\ y_1 + y_2 - y_3 = v_3, \\ y_1 - y_2 - y_3 = v_4, \\ y_1 = v_5, \\ y_2 = v_6, \\ y_3 = v_7. \end{cases}$$

We begin with $p = 2$. Clearly we can not have $v_5 = v_6 = v_7 \neq 0$, otherwise \mathbf{v} will have 4 different nonzero components. Obviously, we cannot have $v_5 = v_6 = v_7 = 0$. If $v_5 = 0$ and v_6 and v_7 are not zero, then v_3 and v_4 must be different and different from v_6 and v_7 . The same happens interchanging v_5 with v_6 or v_7 .

For $v_5 = v_6 = 0$ we have a solution producing an inequality with $h = 1$.

If $v_6 = v_7 = 0$, the system is solvable for $v_1 = v_2 = v_3 = v_4 = v_5 = 1$ and the solution is $y_1 = 1, y_2 = y_3 = 0$; but in this case $B_1 = B_2 = 0$.

Thus, \mathbf{v} should have more than 2 different nonzero components, but in these cases at least 3 coefficients $B_\ell, \ell \geq 1$, are different from zero.

Consider now the following functional equation (where S is again \mathbb{R} or \mathbb{C}):

$$F\left(\sum_{i=1}^5 x_i\right) - \sum_{k=1}^5 F\left(\sum_{i=1}^5 \epsilon(i, k)x_i\right) + 4 \sum_{i=1}^5 F(x_i) = 0, \quad (10)$$

where $\epsilon(i, k)$ is 1 for $i \neq k$ and is -1 for $i = k$. The system (8) becomes

$$\begin{cases} y_1 + y_2 + y_3 + y_4 + y_5 = v_1, \\ -y_1 + y_2 + y_3 + y_4 + y_5 = v_2, \\ y_1 - y_2 + y_3 + y_4 + y_5 = v_3, \\ y_1 + y_2 - y_3 + y_4 + y_5 = v_4, \\ y_1 + y_2 + y_3 - y_4 + y_5 = v_5, \\ y_1 + y_2 + y_3 + y_4 - y_5 = v_6, \\ y_1 = v_7, \\ y_2 = v_8, \\ y_3 = v_9, \\ y_4 = v_{10}, \\ y_5 = v_{11} \end{cases}$$

and it is solvable for $\mathbf{v} = [2, 0, 0, 2, 2, 2, 1, 1, 0, 0, 0]$ and $\mathbf{v} = [3, 1, 1, 1, 3, 3, 1, 1, 1, 0, 0]$ and the solutions are $\mathbf{y} = [1, 1, 0, 0, 0]$ and $\mathbf{y} = [1, 1, 1, 0, 0]$, respectively. It is easy to see that, thanks to the symmetry of the variables, they are the only solutions with $p = 2$. In the two cases, the inequality (6) assume the forms

$$\left| \frac{1}{4}f(2z) - f(z) \right| \leq \frac{\delta}{8} + \frac{3}{4}|f(0)| \leq \frac{11}{64}\delta \quad \text{and} \quad \left| \frac{1}{9}f(3z) - f(z) \right| \leq \frac{\delta}{9} + \frac{8}{9}|f(0)| \leq \frac{\delta}{6}.$$

Another solution for $p = 3$ is obtained with $\mathbf{v} = [4, 4, 2, 2, 2, 2, 1, 1, 1, 1, 0]$ and gives the inequality

$$\left| \frac{1}{4}f(2z) - f(z) \right| \leq \frac{\delta}{16} + \frac{|f(0)|}{4} \leq \frac{5}{64}\delta.$$

Thus, we get the same solution F of the functional equation satisfying the inequalities

$$|F(z) - f(z)| \leq \frac{11}{48}\delta, \quad |F(z) - f(z)| \leq \frac{3}{16}\delta \quad \text{and} \quad |F(z) - f(z)| \leq \frac{5}{48}\delta.$$

Thus, the lowest bound is $\frac{5}{48}\delta$.

The last consideration opens a natural question about the *optimal* bound.

When the system (8) is not solvable under the required conditions, we may proceed as follows.

Given p different nonzero complex numbers $\gamma_1, \gamma_2, \dots, \gamma_p$, we say that inequalities of the forms

$$\left| \sum_{j=1}^p M_j^{(i)} f(\gamma_j w) \right| \leq K_i, \quad i = 1, 2, \dots, r,$$

are independent if the vectors $[M_1^{(i)}, \dots, M_p^{(i)}], i = 1, 2, \dots, r$, are linearly independent.

For a vector \mathbf{v} , we denote by $R(\mathbf{v})$ the set of the nonzero values assumed by its components.

Let $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n-1)} \in \mathbb{K}^s$ satisfying the following conditions:

- for each $k = 1, 2, \dots, n-1$ there exists a family $\{S_\ell^{(k)}\}, \ell = 1, 2, \dots, p_k, n \leq p_k \leq s$, of nonempty disjoint subsets of $\{1, 2, \dots, s\}$ and p_k different nonzero elements of \mathbb{K} , $\gamma_1^{(k)}, \gamma_2^{(k)}, \dots, \gamma_{p_k}^{(k)}$ such that for $i \in S_\ell^{(k)}, v_i^{(k)} = \gamma_\ell^{(k)}$;
- if $\bigcup_{\ell=1}^{p_k} S_\ell^{(k)} \neq \{1, 2, \dots, s\}$, for $i \in S_0^{(k)} := \{1, 2, \dots, s\} \setminus \bigcup_{\ell=1}^{p_k} S_\ell^{(k)}, v_i^{(k)} = 0$; set $B_0^{(k)} := \sum_{i \in S_0^{(k)}} b_i$;
- for each $k = 1, 2, \dots, n-1$ there are n indices $r_1^{(k)}, r_2^{(k)}, \dots, r_n^{(k)} \in \{1, 2, \dots, p_k\}$ such that the sets $\{\gamma_{r_1^{(k)}}^{(k)}, \dots, \gamma_{r_n^{(k)}}^{(k)}\}$ coincide;
- for each $k = 1, 2, \dots, n-1, B_t^{(k)} := \sum_{i \in S_t^{(k)}} b_i = 0$, if $t \notin \{r_1^{(k)}, r_2^{(k)}, \dots, r_n^{(k)}\}$.

We assume that $r_1^{(k)} = 1, r_2^{(k)} = 2, \dots, r_n^{(k)} = n$ for $k = 1, 2, \dots, n-1$.

Theorem 1. Let $n \geq 2$ be the minimum integer for which there are $n-1$ different vectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \dots, \mathbf{v}^{(n-1)}$ having the properties previously stated, and such that the systems

$$\mathbf{A}\mathbf{y} = \mathbf{v}^{(k)}, \quad k = 1, 2, \dots, n-1,$$

are solvable in \mathbb{K}^s with corresponding solutions $\mathbf{y}^{(k)} \in \mathbb{K}^s, k = 1, 2, \dots, n-1$. Suppose that the substitution in the inequality (5) of \mathbf{x} with $w\mathbf{y}^{(k)}, k = 1, 2, \dots, n-1, w \in S$, gives rise to $n-1$ independent inequalities.

Then we obtain the inequality

$$|h_1 f(z) - h_2 f(gz)| \leq \bar{\delta} \tag{11}$$

for certain complex numbers $h_1 \neq 0$ and $h_2, g \in \mathbb{K} \setminus \{0\}$ and a certain positive $\bar{\delta}$.

Proof. If $n = 2$, there is nothing to prove. Assume $n > 2$ and set $\sigma_1 = \gamma_1^{(k)}, \sigma_2 = \gamma_2^{(k)}, \dots, \sigma_n = \gamma_n^{(k)}$. By substituting in the inequality (5) \mathbf{x} with $w\mathbf{y}^{(k)}, k = 1, 2, \dots, n-1$, we obtain the following $n-1$ inequalities:

$$\left| \sum_{i=1}^n B_i^{(k)} f(\sigma_i w) \right| \leq \delta + |B_0^{(k)} f(0)| =: \delta^{(k)}, \quad k = 1, 2, \dots, n-1. \tag{12_k}$$

The condition of independence of the inequalities (12_k) means that the vectors $[B_1^{(k)}, B_2^{(k)}, \dots, B_n^{(k)}]$, $k = 1, 2, \dots, n-1$, are linearly independent.

Moreover, by the minimality of n , for each $i = 1, 2, \dots, n$ at least 2 coefficients $B_i^{(k)}$ are different from zero.

Now we fix our attention on the term $f(\sigma_n w)$. We may assume that $B_n^{(1)} \neq 0$. By using the inequalities (12_1) we eliminate the term $B_n^{(k)} f(\sigma_n w)$ in all other inequalities having $B_n^{(k)} \neq 0$. The inequality obtained from (12_1) and, say, (12_q) is the following:

$$\left| \sum_{i=1}^{n-1} \frac{B_i^{(1)}}{B_n^{(1)}} f(\sigma_i w) - \sum_{i=1}^{n-1} \frac{B_i^{(q)}}{B_n^{(q)}} f(\sigma_i w) \right| = \left| \sum_{i=1}^{n-1} \left(\frac{B_i^{(1)}}{B_n^{(1)}} - \frac{B_i^{(q)}}{B_n^{(q)}} \right) f(\sigma_i w) \right| \leq \frac{\delta^{(1)}}{|B_n^{(1)}|} + \frac{\delta^{(q)}}{|B_n^{(q)}|}.$$

Thus, by keeping together these new inequalities and those originally having $B_n^{(k)} \neq 0$, we get $n-2$ inequalities of the form

$$\left| \sum_{i=1}^{n-1} C_i^{(k)} f(\sigma_i w) \right| \leq \Delta^{(k)}, \quad k = 2, \dots, n-1 \quad (13_k)$$

A simple (but tedious) computation shows that the independence of the inequalities (12_k) implies that of the new inequalities (13_k) , $k = 2, \dots, n-1$.

If for some $i = i_0$ at most one of the coefficients $C_{i_0}^{(k)}$, $k = 2, \dots, n-1$, is different from zero, say $C_{i_0}^{(k_0)}$, then we consider only the $n-3$ inequalities for $k = 2, \dots, n-1$, $k \neq k_0$. These are $n-3$ inequalities containing the same $n-2$ terms in $f(\sigma_i w)$, $i = 1, 2, \dots, n-1$, $i \neq i_0$, and obviously are still independent. Thus, we reduce again to the original situation but with a number d of inequalities less or equal to $n-2$ in $d-1$ terms. By continuing the procedure above, after no more than $n-2$ steps we arrive to an inequality of the form

$$|h_1 f(\sigma^* w) - h_2 f(\sigma^{**} w)| \leq \bar{\delta},$$

where $h_1 \neq 0$, $\sigma^*, \sigma^{**} \in \mathbb{K} \setminus \{0\}$. By setting $z = \sigma^* w$ and $g = \frac{\sigma^{**}}{\sigma^*}$ we get the desired inequality. \square

If h_2 is zero, then f is bounded. Otherwise, if $|\frac{h_1}{h_2}| \neq 1$, we obtain the stability.

Remark 1. The elimination procedure described in the proof of Theorem 1 is worked out by doing some arbitrary choices: which inequalities use and in which order. Different choices may produce different final inequalities.

Going back to the functional equation (8), we easily see that the systems $A\mathbf{y} = \mathbf{v}^{(i)}$, $i = 1, 2$, with $\mathbf{v}^{(1)} = [3, 1, 1, -1, 1, 1, 1]$ and $\mathbf{v}^{(2)} = [1, -1, 3, 1, 1, 1, -1]$ are solvable and $\mathbf{y}^{(1)} = [1, 1, 1]$, $\mathbf{y}^{(2)} = [1, 1, -1]$. Hence, we get the two inequalities

$$|10f(z) - f(3z) - f(-z)| \leq \delta, \quad |6f(z) - f(3z) + 3f(-z)| \leq \delta. \quad (14)$$

By eliminating $f(-z)$, we arrive to

$$\left| f(z) - \frac{1}{9} f(3z) \right| \leq \frac{\delta}{9}.$$

If we eliminate $f(3z)$ from the inequalities (14), we get

$$|f(z) - f(-z)| \leq \frac{\delta}{2}$$

and we cannot proceed.

By eliminating $f(z)$ we obtain

$$\left| f(-z) - \frac{1}{9}f(3z) \right| \leq \frac{4}{9}\delta$$

and changing z into $-z$,

$$\left| f(z) - \frac{1}{9}f(-3z) \right| \leq \frac{4}{9}\delta.$$

By applying the standard method, we obtain a function F , solution of the functional equation, such that

$$|F(z) - f(z)| \leq \frac{\delta}{8} \quad \text{and} \quad |F(z) - f(z)| \leq \frac{\delta}{2}.$$

Thus, again, we have two different bounds, but in this case we know that $\frac{\delta}{8}$ is the optimal one.

4. Final remarks and open problems

1. As we have seen in the previous section, different procedures permit to obtain an inequality of the form (6) from the original inequality (5). In general the relevant parameters appearing in (6), say h and $\bar{\delta}(\delta, |f(0)|)$, are different depending on the way we arrive to (6). When we have stability, that is $|h| \neq 1$, this fact has no influence on the function F , solution of Eq. (4), which we obtain. What changes is the bound we obtain for the distance between f and F . We have $|f(0)| \leq \frac{\delta}{|B|}$ and the right-hand side of the inequality (6) can be majorized by $\bar{\delta}(\delta, \frac{\delta}{|B|})$, that is, by a quantity independent from the function f . Moreover, it is immediately seen that

$$|F(z) - f(z)| \geq \frac{\delta}{|B|}.$$

In general we do not know if this minimum is attained for *any* function f satisfying the inequality (5).

2. If we eliminate the assumption that $B := \sum_{i=1}^s b_i \neq 0$, then every constant function is solution of the Eq. (2). So, if we add any constant function to the function f satisfying the inequality (3), we have again a function satisfying the same inequality and, in case the whole procedure works, producing the same solution of (2). Thus to look for the bounds for the distance between f and F has no sense.

3. As written in the Introduction, δ has been taken as a positive constant only for sake of simplicity in the presentation. In general the right-hand side of the inequality (3) is a function $\delta(x_1, x_2, \dots, x_m) = \delta(\mathbf{x})$. Thus, when we obtain the inequality (6), its right-hand side will be a function $\bar{\delta}(z)$. The conditions giving stability (see [10]) are the convergence of the series

$$\sum_{j=0}^{\infty} |h|^j \bar{\delta}(g^j z)$$

for each z and that

$$\lim_{n \rightarrow +\infty} |h|^n \delta(g^n \mathbf{x}) = 0$$

for each \mathbf{x} .

4. The assumption that S is a vector space over a field \mathbb{K} of characteristic zero has been made to avoid useless complications. We can consider other possibilities, for instance that S is a commutative group or semigroup. In these cases \mathbb{K} becomes \mathbb{Z} or \mathbb{N} . Hence, the conditions of solvability of the system (8) become more restrictive, as seen by the following example.

Consider the functional equation

$$F(x_1 + 3x_2) + F(x_1 - 3x_2) - 2F(x_1) - 2F(3x_2) = 0$$

on $S = \mathbb{Z}$.

The system (8) becomes the following:

$$\begin{cases} y_1 + 3y_2 = v_1, \\ y_1 - 3y_2 = v_2, \\ y_1 = v_3, \\ 3y_2 = v_4. \end{cases}$$

The only solution in \mathbb{R}^2 , under the conditions stated, is $[1, \frac{1}{3}]$, solution which is not in \mathbb{Z}^2 . In this case the nonsolvability of (8) depends on the domain.

Also when the system (8) is solvable under the prescribed conditions, or we can apply Theorem 1, we arrive to an inequality of the form

$$|h_1 f(\sigma^* w) - h_2 f(\sigma^{**} w)| \leq \bar{\delta}$$

and if both σ^* and σ^{**} are different from 1, this inequality says something about the values of f only on a subgroup (subsemigroup) of S .

5. Assume that $s = m + 1$ and that the matrix A has rank m ; obviously in this case we can solve the system (8). This is a very simple sufficient condition. It would be interesting to produce other conditions which guarantee either the solvability of (8) or the possibility of applying Theorem 1. These conditions should be independent from the coefficients b_i s, that is depending only from the arguments in the function F .

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